**Orthogonal Trajectories:** When all the curves in a family $G(x, y, c_1) = 0$ intersect orthogonally all the curves in another family $H(x, y, c_2) = 0$, the families are said to be orthogonal trajectories of each other.

$dy/dx = f(x, y)$ is the differential equation of one family, then the differential equation for the orthogonal trajectories of this family is $dy/dx = -1/f(x, y)$

Example

Find the orthogonal trajectories of $c_1x^2 + y^2 = 1$

If $G(x, y, c_1) = 0$ represents a family of curves, then $dy/dx = f(x, y)$ where $f(x, y) = -\frac{\partial G}{\partial x} / \frac{\partial G}{\partial y}$ represents the slope of the family of curves $G(x, y, c_1)$.

$dy/dx = -\frac{2c_1x}{2y} = -\frac{c_1x}{y}$ where $c_1 = \frac{1-y^2}{x^2}$ (solved $c_1x^2 + y^2 = 1$ for $c_1$) therefore

$$dy/dx = -\frac{(1-y^2)x^2}{y^3} = -\frac{(1-y^2)}{xy}$$

We are looking for a family of curves orthogonal to $G$ which means that the slope of these curves will be equal to the negative reciprocal of the slope of the original family.

For $H(x, y, c_2)$

$$dy/dx = \frac{xy}{1-y^2}$$

Solving this new (separable) differential equation will yield a family of curves $H(x, y, c_2)$ orthogonal to the original family $G(x, y, c_2)$.

Separating variables and integrating yields

$$\int \frac{1-y^2}{y} dy = \int x dx$$

$$\ln|y| - \frac{1}{2}y^2 = \frac{1}{2}x^2 + c_2$$
Mixing Problems

Let \( x(t) \) = the amount of substance in the tank at any given time \( t \).

\[
\frac{dx}{dt} = \left[ \text{rate in} \right] - \left[ \text{rate out} \right]
\]

\[
\frac{dx}{dt} = \left[ \frac{a \text{ units of substance}}{\text{volume of solution}} \cdot \frac{b \text{ volume}}{\text{time}} \right] - \left[ \frac{x(t) \text{ units}}{f + (b-c) \text{ volume}} \cdot \frac{c \text{ volume}}{\text{time}} \right]
\]

where \( x(0) = g \)

Example

A brine solution with a concentration of .8kg/gal is pumped into a tank at a rate of 6 gallons per minute. The tank initially holds 500 kgs of salt dissolved in 1000 gallons of solution. What is the amount of salt in the tank after \( t \) minutes if the solution is also leaking out of the tank at a rate of 0.2 gal/min?

\[
\frac{dx}{dt} = \left[ \frac{8 \text{ kg}}{\text{gal}} \cdot \frac{6 \text{ gal}}{\text{min}} \right] - \left[ \frac{x(t)}{1000 + (6-0.2)t} \cdot \frac{0.2 \text{ gal}}{\text{min}} \right]; \quad x(0) = 500
\]

\[
\frac{dx}{dt} = 4.8 - \frac{0.2x}{1000 + 5.8t} \quad \text{or} \quad \frac{dx}{dt} + \frac{0.2}{1000 + 5.8t} x = 4.8 \quad x(0) = 500
\]

Solving:

\[
e^{0.2 \int_{1000+5.8t}^{1000+5.8t} dt} = e^{0.0345 \ln |1000 + 5.8t|} = (1000 + 5.8t)^{0.0345}
\]

\[
x(1000 + 5.8t)^{0.0345} = 4.8 \int (1000 + 5.8t)^{0.0345} dt
\]

\[
x = 0.8(1000 + 5.8t) + (1000 + 5.8t)^{-0.0345} c
\]

For \( x(0) = 500 \) \( c = -380.73 \)

So \( x = 0.8(1000 + 5.8t) - 380.73(1000 + 5.8t)^{-0.0345} \)
Spring Problems

A mass \( m \) is placed on a spring stretching it a distance \( x_1 \). The entire spring-mass system is then immersed in a substance giving the system a damping force with a damping constant \( \beta \). The motion of the system is driven by an external force \( E(t) \). If the spring is stretched initially by a distance \( s_0 \) and then released with initial velocity \( v_0 \), set up a DEQ that models this system.

**DEQ:** \( m x'' + \beta x' + kx = E(t); \quad x(0) = s_0, \ x'(0) = v_0 \)

where \( k \) is the spring constant and if not given can be calculated using \( k = \frac{mg}{x_1} \) by Hooke’s Law (Note: \( g \) is the gravitational constant and on Earth \( g \approx 9.8 \text{ N/kg} \))

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**Example**

A 2-kilogram mass is attached to a spring whose constant is 16 n/m, and the entire system is then submerged in a liquid that imparts a damping force numerically equal to 10 times the instantaneous velocity. Determine the equation of motion if the mass is released 1 meter below the equilibrium position with a downward velocity of 12 m/s.

Write out what is given:

\[
\begin{align*}
m &= 2 \text{ kg} \quad & v_0 &= 12 \text{ m/s} \\
\beta &= 10 \text{ N/(m/s)} \quad & s_0 &= 1 \text{ m} \\
k &= 16 \text{ N/kg} \quad & \text{ } \text{ } \\
E(t) &= 0 \\
\end{align*}
\]

Write out the DEQ:

\( 2x'' + 10x' + 16x = 0 \); \( x(0) = 1 \), \( x'(0) = 12 \)

Solve DEQ:

\[
2m^2 + 10m + 16 = 0
\]

\[
m = \frac{-10 \pm \sqrt{28}}{2} = -5 \pm i
\]
Solution: 
\[ y = e^{-5t} [c_1 \cos(\sqrt{7}t) + c_2 \sin(\sqrt{7}t)] \]

since \( x(0) = 1, \quad c_1 = 1 \)
and \( x'(0) = 12, \quad c_2 = \frac{17}{\sqrt{7}} \)

**Note:** Using the following trig relationships we can represent the solution as a single term.

For a solution of the form \( y = e^{at}[c_1 \cos(bt) + c_2 \sin(bt)] \)
let \( A^2 = c_1^2 + c_2^2 \) and \( \Phi = \tan^{-1}(c_1/c_2) \)

Then \( y = e^{at}[A \sin(bt + \Phi)] \)

Solution in single term form: 
\[ y = e^{-5t}[A \sin(\sqrt{7}t + \Phi)] \]

where \( A = \sqrt{1^2 + \left(\frac{17}{\sqrt{7}}\right)^2} = \frac{2\sqrt{518}}{7} = 6.5027 \)
and \( \Phi = \tan^{-1}\left(\frac{\sqrt{7}}{17}\right) = 0.15439 \)