EXACT EQUATIONS AND INTEGRATING FACTORS

First-order Differential Equations for which we can find exact solutions
Study the patterns carefully. The first step of any solution is correct identification of the type of differential equation.

Total Differential of a Function \( F(x,y) \)
\( F \) is a function of two variables which has continuous partial derivatives over a domain \( D \).
The total differential of \( F \) is defined as :

\[
dF(x,y) = \frac{\partial F(x,y)}{\partial x} \, dx + \frac{\partial F(x,y)}{\partial y} \, dy \quad \forall (x,y) \in D
\]

Exact Differential
\( M(x,y)dx + N(x,y)dy \) is an exact differential over \( D \) if \( \exists F(x,y) \) such that

\[
M(x,y)dx + N(x,y)dy = dF(x,y)
\]

In other words, \( M(x,y) = \frac{\partial F(x,y)}{\partial x} \) and \( N(x,y) = \frac{\partial F(x,y)}{\partial y} \).

Exact Differential Equation
\( M(x,y)dx + N(x,y)dy = 0 \) is an exact differential equation if 
\( M(x,y)dx + N(x,y)dy \) is an exact differential.

Given a DE that can be written in the form \( M(x,y)dx + N(x,y)dy = 0 \),
find \( F(x,y) \) such that \( dF(x,y) = M(x,y)dx + N(x,y)dy \).

The following theorem supplies a method for determining whether or not a DE is exact.

Theorem 2.1
M(x,y) and N(x,y) are functions with continuous first partial derivatives in the domain D. M(x,y) and N(x,y) form the DE

\[ M(x,y)dx + N(x,y)dy = 0 \]

The above DE is exact in D if and only if

\[ \frac{\partial}{\partial y} M(x,y) = \frac{\partial}{\partial x} N(x,y) \quad \forall (x,y) \in D \]

**Solution Method for Exact Equations**

1. Show \( \frac{\partial}{\partial y} M(x,y)dx = \frac{\partial}{\partial x} N(x,y)dy \)

2. Set \( \frac{\partial F(x,y)}{\partial x} = M(x,y) \). Integrate with respect to x to get \( F(x,y) \)

\[ F(x,y) = \int M(x,y)dx + \phi(y) \]

3. Differentiate with respect to y to get \( N(x,y) \)

\[ \frac{\partial}{\partial y} \left[ \int M(x,y)dx + \phi(y) \right] = N(x,y) \]

4. Solve for \( \phi(y) \)

The answer will be of the form : \( F(x,y) = g(x,y) + \phi(y) \), if no boundary value is given.

**Exercises:** Which of these equations are exact?

a. \( (y^2 + 3)dx + (2xy - 4)dy = 0 \)

b. \( (3x^2y + 2)dx - (x^3 + y)dy= 0 \)

c. \( (\theta^2 + 1)\cos rd\theta + 2\theta \sin rd\theta = 0 \)

d. \( \left(\frac{x}{y^2} + x\right)dx + \left(\frac{x^2}{y^3} + y\right)dy = 0 \)

e. \( \left(\frac{2y^{3/2} + 1}{x^{1/2}}\right)dx + (3x^{1/2} y^{1/2} - 1)dy = 0 \)
Exercises: Solve

a. \( (3x^2y^2 - y^3 + 2x)\,dx + (2x^3y - 3xy^2 + 1)\,dy = 0 \) given \( y(-2) = 1 \)

b. \( (ye^x + 2e^x + y^2)\,dx + (e^x + 2xy)\,dy = 0 \)

c. Find \( A \) so that this equation is exact. \( (Ax^2y^2 + y^2)\,dx + (x^3 + 4xy)\,dy = 0 \)

Integrating Factors

What if \( \frac{\partial M(x,y)}{\partial y} \neq \frac{\partial N(x,y)}{\partial x} \)

We may be able to rewrite the equation so that it is exact.

Definition

If \( M(x,y)\,dx + N(x,y)\,dy = 0 \) is not exact, but \( \mu(x,y)M(x,y)\,dx + \mu(x,y)N(x,y) = 0 \) is exact, then \( \mu(x,y) \) is called an integrating factor.

Example

Show that \( (y^2 + 2xy)\,dx - x^2\,dy = 0 \) is not exact, then find \( n \) such that \( y^n \) is an integrating factor.

i. \( \frac{\partial}{\partial y}(y^2 + 2xy) = 2y + 2x \neq \frac{\partial}{\partial x}(-x^2) = -2x \)

therefore the DE is not exact.

ii. Multiply the DE by \( y^n \), then solve.

\( y^n(y^2 + 2xy)\,dx - y^n x^2 \,dy = 0 \)

\( \frac{\partial}{\partial y}(y^{n+2} + 2xy^{n+1}) = (n + 2)y^{n-1} + 2(n + 1)xy^n \) must equal

\( \frac{\partial}{\partial x}(-y^n x^2) = -2y^n x \) which means \( (n + 2)y^{n+1} \) must equal 0 and

\( 2(n + 1)xy^n \) must equal \( -2xy^n \) for this to be so \( n \) must equal \(-2\)

iii. Now, solve the equation.
\[ y^{-2}(y^2 + 2xy)dx - y^{-2}(x^2)dy = 0 \]
\[ (1 + 2xy^{-1})dx + (-x^2y^2)dy = 0 \]

**Solution:**
\[ F(x, y) = x + x^2y^{-1} + c \]

**Review:**

i) What is a linear DE?

ii) What is a first order DE?

iii) What is an exact DE of order 1?

iv) What is an integrating factor?

**Exercise:**

a. Examine \( 4xydx + (x^2 + 1)dy = 0 \). Is it linear? order 1? exact?

b. Multiply the given DE by \( \frac{1}{y(x^2 + 1)} \) and answer the same questions about the result.

**Separable Equations**

\[ F(x)G(y)dx + f(x)g(y)dy = 0 \]

This type of DE is called **separable** because it can be written in the form (variables can be separated)

\[ M(x)dx + N(y)dy = 0 \]

The first equation is usually not exact but multiplying it by the appropriate integrating factor will make it exact, but use of an integrating factor may eliminate solutions or may lead to extraneous solutions.

After multiplying by the integrating factor \( \frac{1}{f(x)G(y)} \) the equation becomes:

\[ \frac{F(x)}{f(x)}dx + \frac{g(y)}{G(y)}dy = 0 \]

where \( M(x) = \frac{F(x)}{f(x)} \) and \( N(x) = \frac{g(y)}{G(y)} \).

Solutions are of the form

\[ \int M(x)dx + \int N(y)dy + c \quad \text{where} \quad f(x) \neq 0 \quad \& \quad G(y) \neq 0. \]
**Exercises:** Which equations are separable? For each separable equation find the correct integrating factor.

a. \((xy + 2x + y + 2)dx + (x^2 + 2x)dy = 0\)  
   Hint: factor \(xy + 2x + y + 2\)

b. \(\csc y \, dx + \sec x \, dy = 0\)

c. \((e^y + 1)\cos u \, du + e^y (\sin u + 1) \, dv = 0\)

d. \((x + y) \, dx - x \, dy = 0\)

e. \(v^3 \, du + (u^3 - uv^2) \, dv = 0\)

**Homogeneous DE**

If \(M(x,y) \, dx + N(x,y) \, dy = 0\) can be written in the form \(\frac{dy}{dx} = f(x,y)\) where \(f(x,y) = g\left(\frac{y}{x}\right)\) then the DE is **homogeneous**. If \(F(tx,ty) = t^n F(x,y)\) then \(F\) is **homogeneous of degree \(n\)**.

A homogeneous equation can be transformed to a separable equation by a change of variable \(y = vx\). The new equation is separable in \(v\) and \(x\), so it can be solved.

Given \(M(x,y) \, dx + N(x,y) \, dy = 0\) is homogeneous, let \(y = vx\). Then \(\frac{dy}{dx} = v + x \frac{dv}{dx}\). Since the given DE is homogeneous, we know it can be written in the form \(\frac{dy}{dx} = g\left(\frac{y}{x}\right)\). Since \(y = vx\),

\[ g\left(\frac{y}{x}\right) = g\left(\frac{vx}{x}\right) = g(v). \]

And since

\[ \frac{dy}{dx} = g\left(\frac{y}{x}\right) \]

\[ v + x \frac{dv}{dx} = g(v). \]

Separating variables \(v \, dx + x \, dv = g(v) \, dx:\) \[ [v - g(v)] \, dx + x \, dv = 0:\] \[ \frac{1}{x} \, dx + \frac{1}{v - g(v)} \, dv = 0 \]

To solve, integrate:

\[ \int \frac{1}{x} \, dx + \int \frac{1}{v - g(v)} \, dv = c \]

\[ \ln|x| + F(v) = c \quad \text{or} \quad \ln|x| + F\left(\frac{y}{x}\right) = c \]
Exercises: Solve

a. \((x + y)dx + (-x)dy = 0\)

b. \(y^3dx + (x^3 - xy^2)dy = 0\)

c. \((y + 2)dx + y(x + 4)dy = 0\) where \(y(-3) = -1\)

A Linear First Order DE with Variable Coefficients can be written in the following form.

\[
\frac{dy}{dx} + P(x)y = Q(x)
\]

This equation is exact only when \(P(x) = 0\).

Proof:

Write DE in the form \(M(x, y)dx + N(x, y)dy = 0\)

\[
\frac{dy}{dx} + P(x)y = Q(x)
\]

\(dy + P(x)ydx = Q(x)dx\)

\(P(x)ydx - Q(x)dx + dy = 0\)

\([P(x)y - Q(x)]dx + dy = 0\)

But \(\frac{\partial}{\partial y} M(x, y) = \frac{\partial}{\partial y} [P(x) - Q(x)] = P(x)\)

\(\frac{\partial}{\partial x} N(x, y) = \frac{\partial}{\partial x} (1) = 0\) \(\therefore P(x)\) must equal 0 for the DE to be exact.
The equation \[ P(x)y - Q(x) \] dx + dy = 0 can be solved through by finding a proper integrating factor. The factor depends only on x, so call it \( \mu(x) \).

The new equation \[ \mu(x)P(x)y - \mu(x)Q(x) \] dx + \( \mu(x) \) dy = 0 is exact so...

since \[ \frac{\partial}{\partial y} [\mu(x)P(x)y - \mu(x)Q(x)] = \mu(x)P(x) \]

\[ \frac{\partial}{\partial x} [\mu(x)] = \frac{d}{dx} \mu(x) \]

\[ \mu(x)(x) = \frac{d}{dx} \mu(x) \text{ or } \mu P(x) = \frac{d \mu}{dx} \]

\[ P(x) dx = \frac{1}{\mu} d \mu \]

\[ \int P(x) dx = \int \frac{1}{\mu} d \mu \]

\[ \int P(x) dx = \ln |\mu| \]

\[ \therefore \mu(x) = e^{\int P(x) dx} \text{ assuming } \mu(x) > 0 \]

The solution of the DE is therefore of the form

\[ y = e^{-\int P(x) dx} \left[ \int e^{\int P(x) dx} Q(x) dx + c \right] \]

Exercise:

a. \[ x^4 \frac{dy}{dx} + 2x^3 y = 1 \]
Bernoulli Equations

An equation of the form \( \frac{dy}{dx} + P(x)y = Q(x)y^n \) is called a Bernoulli Equation.

Bernoulli equations can be transformed to linear equations by \( v = y^{1-n} \).

Proof:

\[
\frac{dy}{dx} + P(x)y = Q(x)y^n
\]

\( (y^n) \) multiply

\[
y^{-n}\frac{dy}{dx} + P(x)y^{1-n} = Q(x)
\]

let \( v = y^{1-n} \)

\[
\frac{dv}{dx} = (1-n) y^{-n} \frac{dy}{dx}
\]

\[
y^{-n} = \frac{1}{(1-n) \frac{dy}{dx}}
\]

\( (1-n) \) multiply

\[
\frac{dv}{dx} + (1-n) P(x)v = (1-n)Q(x)
\]

let \( P_1(x) = (1-n)P(x) \)

\[
\frac{dv}{dx} + P_1(x)v = Q_1(x)
\]

\( Q_1(x) = (1-n)Q(x) \)

which is linear in \( v \)!
Example: \[ (x^2 + 1) \frac{dy}{dx} + 4xy = 3y \]

change to Bernoulli \[ \frac{dy}{dx} + \frac{4x}{x^2 + 1} y = \frac{3}{x^2 + 1} y \]

where \( P(x) = \frac{4x}{x^2 + 1} \) & \( Q(x) = \frac{3}{x^2 + 1} \)

since \( y^n = y^1 \)

\( y^1 \) multiply \[ y^{-1} \frac{dy}{dx} + \frac{4x}{x^2 + 1} = \frac{3}{x^2 + 1} \]

Let \( v = y^{1-n} = y^{1-(1)} = y^2 \) solving for \( y = v^{1/2} \)

\[ \frac{dv}{dx} = 2y \frac{dy}{dx} \]

solving for \[ \frac{dy}{dx} = \frac{1}{2y} \frac{dv}{dx} \]

\[ v^{-1/2} \left( \frac{1}{2} v^{-1/2} \right) \frac{dv}{dx} + \frac{4x}{x^2 + 1} = \frac{3}{x^2 + 1} \]

separate and integrate \[ \int \frac{1}{2v} dv = \int \frac{3-4x}{x^2 + 1} dx \]

\[ \frac{1}{2} \int \frac{1}{v} dv = \ln|v|^{1/2} = \ln y = \int \frac{3-4x}{x^2 + 1} dx \]

\[ = -2 \int \frac{2x}{x^2 + 1} + 3 \int \frac{1}{x^2 + 1} dx \]

\[ \ln y = -2 \ln|x^2 + 1| + 3 \arctan x \]

solving for \( y \)

\[ y = \frac{e^{3\arctan x}}{(x^2 + 1)^{1/2}} \]

Exercise:
a. \[ \frac{dy}{dx} + \frac{y}{x^2} = \frac{1}{x^2} \]