Second Order DEQ with Constant Coefficients

For a 2\textsuperscript{nd} order DEQ $a y'' + b y' + c y = f(x)$

1. **Finding the solution of a homogeneous equation:** $ay'' + by' + cy = 0$

   a) Assume solutions of the form $y = e^{mx}$.
   b) Substitute $y, y' = me^{mx}$, and $y'' = m^2 e^{mx}$ into the DEQ.

   $$e^{mx}(am^2 + bm + c) = 0$$

   c) Solve the resulting characteristic equation $am^2 + bm + c = 0$ for $m$.

   $$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

   This yields three possible types of solutions:

   i) If $m$ is two distinct numbers then, $y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$
   ii) If $m$ is a repeated solution then, $y = c_1 e^{mx} + c_2 xe^{mx}$
   iii) If $m$ is a complex number $\alpha \pm \beta i$ then, $y = e^{\alpha x} (c_1 \cos \beta x + c_2 \sin \beta x)$

-----Examples

1. Solve: $y'' - 3y' + 2y = 0$

   Characteristic equation: $m^2 - 3m + 2 = 0$
   $$m = \frac{3 \pm \sqrt{9 - 8}}{2} = 2, 1$$
   Solution: $y = c_1 e^{2x} + c_2 e^{x}$

2. Solve: $y'' - 4y' + 4y = 0$

   Characteristic equation: $m^2 - 4m + 4 = 0$
   $$m = \frac{4 \pm \sqrt{16 - 16}}{2} = 2$$
   Solution: $y = c_1 e^{2x} + c_2 xe^{2x}$

3. Solve: $2y'' + 2y' + y = 0$

   Characteristic equation: $2m^2 + 2m + 1 = 0$

   $$m = \frac{-2 \pm \sqrt{4 - 4(2)(1)}}{2(2)} = -\frac{1}{2} \pm \frac{1}{2} i$$

   Solution: $y = e^{-\frac{1}{2} x} [c_1 \cos \frac{1}{2} x + c_2 \sin \frac{1}{2} x]$
Finding the solution of a non-homogeneous equation: \( a_1 y'' + a_2 y' + a_3 y = f(x) \)

Method: Undetermined Coefficients

a) First find the solution of the homogeneous equation
\[ a_1 y'' + a_2 y' + a_3 y = 0 \]

b) Use the following to identify possible terms of a particular solution in terms of \( f(x) \).

<table>
<thead>
<tr>
<th>Terms in ( f(x) )</th>
<th>Terms in Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x^n )</td>
<td>( c_3 + c_4 x + c_5 x^n + \ldots + c_{n+3}x^n )</td>
</tr>
<tr>
<td>( \cos(bx+c) ) or ( \sin(bx+c) )</td>
<td>( c_3\cos(bx+c) + c_4\sin(bx+c) )</td>
</tr>
<tr>
<td>( e^{ax} )</td>
<td>( c_3e^{ax} )</td>
</tr>
</tbody>
</table>

Or a combination of these functions

Note: You may need to revise the terms that are already solutions of the homogeneous case by multiplying by \( x \) to the appropriate power.

c) Form a particular solution \( y_p \) using the appropriate set of possible terms with undetermined coefficients.

d) Differentiate \( y_p \) and sub into the original DEQ, then solve for the undetermined coefficients.

-----Example

1. Solve \( y'' - 3y' + 2y = 2x^2 - 3e^{2x} + 3\cos x \)

The solution for the homogeneous equation is \( y_c = c_1 e^x + c_2 e^{2x} \)

Set of possible terms for \( y_p \) is \( \{x^2, x, 1, e^{2x}, \cos x, \sin x\} \) but \( e^{2x} \) is a solution of the homogeneous case so the revised set of terms is \( \{x^2, x, 1, xe^{2x}, \cos x, \sin x\} \).

Therefore the particular solution with undetermined coefficients is

\[ y_p = c_3 x^2 + c_4 x + c_5 + c_6 xe^{2x} + c_7 \cos x + c_8 \sin x \]

The derivatives are

\[ y_p' = 2c_3 x + c_4 + c_6 e^{2x} + 2c_6 xe^{2x} - c_7 \sin x + c_8 \cos x \]
\[ y_p'' = 2c_3 + 4c_6 e^{2x} + 4c_6 xe^{2x} - c_7 \cos x - c_8 \sin x \]

Subbing in to the DEQ yields

\[ 2c_3 + 4c_6 e^{2x} + 4c_6 xe^{2x} - c_7 \cos x - c_8 \sin x - 3(2c_3 x + c_4 + c_6 e^{2x} + 2c_6 xe^{2x} - c_7 \sin x + c_8 \cos x) + 2(c_3 x^2 + c_4 x + c_5 + c_6 xe^{2x} + c_7 \cos x + c_8 \sin x) = 2x^2 - 3e^{2x} + 3\cos x \]
Regrouping terms yields

\[(2c_3 - 3c_4 + 2c_5) + (-6c_3 + 2c_4)x + (2c_3)x^2 + (c_6)e^{2x} + (c_7 - 3c_8)\cos x + (c_8 + 3c_7)\sin x = 2x^2 - 3e^{2x} + 3\cos x\]

Coefficients of terms on the left must equal coefficients of the corresponding terms on the right.

\[
\begin{align*}
2c_3 - 3c_4 + 2c_5 &= 0 \\
-6c_3 + 2c_4 &= 0 \\
2c_3 &= 2 \\
c_6 &= -3 \\
c_7 - 3c_8 &= 3 \\
c_8 + 3c_7 &= 0
\end{align*}
\]

Solving yields: \(c_3 = 1, \ c_4 = 3, \ c_5 = 7/2, \ c_6 = -3, \ c_7 = 3/10, \ c_8 = -9/10\)

so \(Y_p = x^2 + 3x + 7/2 - xe^{2x} + 3/10 \cos x - 9/10 \sin x\)

Therefore the general solution is \(y = y_c + y_p\)

\[y = c_1e^x + c_2e^{2x} + x^2 + 3x + 7/2 - 3xe^{2x} + 3/10 \cos x - 9/10 \sin x\]

**Method: Differential Operators (Annihilators)**

a) A differential operator \(D^{(n)}\) transforms \(y = f(x)\) into the \(n\)th derivative of \(y\) (ie \(y^{(n)}\))

Example:

\[
\begin{align*}
D^0y &= y \\
D^1y &= y' \\
D^2y &= y'' \\
&\text{etc.}
\end{align*}
\]

b) An annihilator is an operator \(L\) composed of differential operators that transforms a function to zero. (ie. \(L f(x) = 0\))

Common Annihilators:

<table>
<thead>
<tr>
<th>(f(x))</th>
<th>Annihilator</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x^0, x, x^2, \ldots, x^n)</td>
<td>(D^{n+1})</td>
</tr>
<tr>
<td>(x^0e^{ax}, xe^{ax}, x^2e^{ax}, \ldots, x^ne^{ax})</td>
<td>((D - \alpha)^{n+1})</td>
</tr>
<tr>
<td>(\cos \beta x \text{ or } \sin \beta x)</td>
<td>((D^2 + \beta^2))</td>
</tr>
<tr>
<td>(x^0e^{ax}\cos \beta x, xe^{ax}\cos \beta x, \ldots, x^ne^{ax}\cos \beta x)</td>
<td>([D^2 - 2\alpha D + (\alpha^2 + \beta^2)]^{n+1})</td>
</tr>
</tbody>
</table>
| \(x^0e^{ax}\sin \beta x, xe^{ax}\sin \beta x, \ldots, x^ne^{ax}\sin \beta x\) | \]
-----Examples

1. Find the annihilator for \( f(x) = 1 + 6x - 2x^3 \)
   \[
   D^4(1 + 6x - 2x^3) = 0 \quad \text{Note: We only need to use the annihilator for the highest power of } x.
   \]

2. Find the annihilator for \( f(x) = x^2 e^{2x} \sin 3x \)
   \[
   (D^2 - 4D + 13)^3(x^2 e^{2x} \sin 3x) = 0 \quad \text{Note: This fits the form } [D^2 - 2\alpha D + (\alpha^2 + \beta^2)]^{n+1}
   \text{ where } n = 2, \alpha = 2, \beta = 3.
   \]

3. Find the annihilator for \( f(x) = x^2 + e^{-3x} + \cos 4x \)
   \[
   (D^3)(D+3)(D^2 + 16)(x^2 + e^{-3x} + \cos 4x) = 0
   \]
   \[
   \text{Note: The annihilator is written as a product of the individual annihilators.}
   \]

4. Solve \( y'' - 3y' + 2y = 2x^2 - 3e^{2x} + 3\cos x \)
   \( (\text{Solved previously using the undetermined coefficient method, now use the annihilator method to solve.}) \)

   a) For the homogeneous solution \( (D^2 - 3D + 2)y = 0 \)
   \[
   (D - 2)(D - 1)y = 0 \quad \text{which yields } D = 2 \text{ or } Dy = 2y \left( \frac{dy}{dx} = 2y \right) \text{ giving } y = c_1 e^{2x}
   \]
   \[
   D = 1 \text{ or } Dy = 1y \left( \frac{dy}{dx} = y \right) \text{ giving } y = c_2 e^x
   \]
   \[
   Y_c = c_1 e^{2x} + c_2 e^x
   \]

   b) The annihilator for \( f(x) = 2x^2 - 3e^{2x} + 3\cos x \) is \( D^3(D - 2)(D^2 + 1)f(x) = 0 \)
   so
   \[
   D^3 = 0 \text{ repeated solution giving } c_1 x^2 + c_4 x + c_5
   \]
   \[
   D - 2 = 0 \text{ giving } c_6 e^{2x}
   \]
   \[
   D^2 + 1 = 0 \text{ giving } c_7 \cos x + c_8 \sin x
   \]
   \[
   Y_p = c_3 x^2 + c_4 x + c_5 + c_6 x e^{2x} + c_7 \cos x + c_8 \sin x
   \]

   c) Now solve \( (D^2 - 3D + 2)y_p = 2x^2 - 3e^{2x} + 3\cos x \) for the undetermined coefficients as in previous method.
   \[
   y = c_1 e^x + c_2 e^{2x} + x^2 + 3x + 7/2 - 3xe^{2x} + 3/10 \cos x - 9/10 \sin x
   \]