Marginal Functions
In Economics

Purpose

Economists look at how costs and benefits change as there are small changes in actions. We call this marginal analysis, and it is perhaps the key concept in economic analysis. It is an acknowledgement that people (should) make a decision based on the incremental gains and losses that result from that decision, and that sunk costs (money, time or other things of worth already expended and unredeemable) should not matter. This handout is designed to assist the student in deriving these marginal costs and revenues, as well as the Elasticity of Demand.

Functions

1. The Total Cost Function, denoted \( C(x) \), gives the total cost to produce \( x \) units of a given product.

   Example 1-A: A company makes fuel tanks for cars. The total weekly cost (in dollars) of producing \( x \) tanks is given by
   \[
   C(x) = 10,000 + 90x - .05x^2
   \]
   The actual cost in producing the 20th unit would be:
   \[
   C(20) - C(19) = (10,000 + 90(20) - .05(20)^2) - (10,000 + 90(19) - .05(19)^2) = $87.95
   \]

2. The derivative of \( C(x) \) is the Marginal Cost Function. The marginal cost function approximates the actual cost of producing an additional item given that production is already in process. \( C'(n-1) \) gives an approximation of the actual cost in producing the \( n \)th unit.

Example 1-B: The actual cost in producing the 21st tank would be
\[
C(21) - C(20) = (10,000 + 90(21) - .05(21)^2) - (10,000 + 90(20) - .05(20)^2) = $87.95
\]
Example 1-C: \( C'(x) = 90 - .1x \). The actual cost of producing the 21\(^{st}\) tank would be: \( 90 - .1(20) = $88.00 \)

This is a very close approximation to the exact amount ($87.95) discovered earlier. Therefore, the Marginal Cost Function is used as an approximation.

3. The Average Cost Function, denoted \( \overline{C}(x) \), is \( \frac{C(x)}{x} \). This gives the average cost of producing \( x \) units of a particular item. The derivative of the Average Cost Function \( \overline{C}'(x) \), is called the Marginal Average Cost Function.

Example 1-C: Consider the cost function: \( C(x) = 400 + 20x \).

The Average Cost Function would be: \( \overline{C}(x) = \frac{400 + 20x}{x} \).

The Marginal Average Cost Function would be: \( \overline{C}'(x) = -\frac{400}{x^2} \).

Revenue Functions:

1. The Revenue Function \( R(x) \) gives the revenue realized by a company from the sale of \( x \) units of a particular item. If \( p \) is the unit selling price function then:

\[ R(x) = px \]

Example 2-A: If \( p = 10 - 0.001x \), then \( R(x) = 10x - 0.001x^2 \). The revenue realized from selling 1000 units would be: \( 10(1000) - 0.001(1000)^2 = $9,000 \)

2. The Marginal Revenue Function is the derivative of the Revenue Function. The marginal revenue function gives an approximation of the actual revenue realized from the sale of an additional unit given that sales are already at a certain level.

Example 2-B: The actual revenue made from the sale of the 30\(^{th}\) unit would be \( R(30) - R(29) \): \([10(30) - 0.001(30)^2] - [10(29) - 0.001(29)^2] = $9.941 \)

Using the Marginal Revenue Function to approximate the revenue for the 30\(^{th}\) unit, we first find that \( R'(x) = 10 - 0.002x \). Then, evaluating \( R'(29) \) we get \( 10 - 0.002(29) = 9.942 \)

This is another very close approximation. This one is so close that they both round to the same cent.
Profit Functions

1. The Profit Function is simply the Revenue Function minus the Cost Function. Consider the previous revenue function: \( R(x) = 10x - 0.001x^2 \). The cost function associated is \( C(x) = 7000 + 2x \). Therefore, the profit function, 
   
   \[ P(x) = R(x) - C(x) \]
   
   would be:
   
   \[ P(x) = (10x - 0.001x^2) - (7000 + 2x) \]
   
   \[ P(x) = -0.001x^2 + 8x - 7000 \]
   
   The relative max of the profit function gives the output level for maximum profit.

Example 3: The graph of \( P(x) \) is shown below. From analysis of the graph, you can see that at output levels of 1000 and 7000 the company breaks even. At outputs above 7000 and below 1000, the company loses money. The maximum profit is achieved at an output level of 4000.

The derivative of the profit function, of course, is the Marginal Profit Function. Again, this can be used to approximate the profit of a unit given that production is already at a certain level.
Elasticity of Demand

1. One would normally expect that by lowering the price of an item the demand for that item would increase, and by raising the price the demand would decrease. To measure the sensitivity of demand against the change in price, we define the Elasticity of Demand denoted $E(p)$. The Elasticity of Demand is the limit of the relative change in demand $x$ divided by the relative change in price $p$.

If $x$ is written as a function of $p$ (price demand equation: $x = f(p)$), then
\[
E(p) = \frac{-pf'(p)}{f(p)}.
\]

If $E(p) > 1$ then the demand is said to be elastic.
If $E(p) = 1$ then the demand has unit elasticity.
If $E(p) < 1$ then the demand is inelastic.

**Example 4-A:** If $x = f(p) = -50p + 20000$, then we determine that $f'(p) = -50$, which means that $E(p) = \frac{-50p}{-50p + 20000} = \frac{p}{400 - p}$.

At a price of $300 per unit we find that $E(300) = \frac{300}{400 - 300} = 3$. Since $E(p) > 1$ the demand is elastic. This tells us that when the price is set at $300, an increase of 1% in the unit price will cause a 3% decrease in the quantity demanded.

**Example 4-B:** If the price is set at $200 per unit we find that $E(200) = \frac{200}{400 - 200} = 1$. Since $E(p) = 1$ the demand has unit elasticity. This tells us that when the price is set at $200, an increase of 1% in the unit price will cause a 1% decrease in the quantity demanded.

**Example 4-C:** If the price is set at $100 per unit we find that $E(100) = \frac{100}{400 - 100} = \frac{1}{3}$. Since $E(p) < 1$ the demand is inelastic. This tells us that when the price is set at $100, an increase of 1% in the unit price will cause only a .33% decrease in the quantity demanded.