OPTIMIZATION

PURPOSE
An essential “real life” application of algebraic principles and differential calculus is the ability to determine the optimum solution for a given equation. This handout is designed to help the student identify the type of optimization required, as well as a method to solve the resultant equation.

PROCEDURE
In general optimization problems want to find the maximum or minimum value a function can be. A suggested procedure is to:

1. Sketch a picture of what is being optimized.
2. Find the function of what is being optimized.
3. Find the constraint function.
4. Solve the constraint function for the variable of interest and replace in the function being optimized.
5. Take the derivative and find the critical value(s) to maximize or minimize the function.
6. Check to ensure the answer is the relative maximum or minimum needed.

EXAMPLE 1: Maximize Area

We need to enclose a rectangular playpen for barn animals with 300 feet of fence. The side of the barn will be one side, so it doesn’t need any fencing. What is the maximum area available?

Step 1: Sketch

Step 2: Since we want to maximize the area:  \( A = XY \)
Step 3: Since we only have 300 feet of fence our constraint is: \(300 = X + 2Y\)

Step 4: We can solve for either \(X\) or \(Y\), but for the sake of familiarity we will solve for \(Y\):

\[Y = \frac{1}{2}X + 150.\]

We then replace \(Y\) in our area equation: 

\[A = X\left(-\frac{1}{2}X + 150\right) \Rightarrow\]

\[A = -\frac{1}{2}X^2 + 150X\]

Step 5: To maximize the equation we need to take the derivative and set it equal to zero to find the critical numbers. The derivative is 

\[A' = -X + 150.\]

After setting it equal to zero and solving we get \(X = 150\).

We use this value to go back and find \(Y\) (constraint equation). We see that \(Y\) is equal to 75. So our dimensions are 150x75, which gives us a maximum area of 11250 sq ft.

Note: If a graph of the area function is plotted the relative max can be found by looking at the graph. The relative max occurs at the point where \(X = 150\) and the value is 11250. This gives us
EXAMPLE 2:  

Minimize Time

An athlete competing in a race must determine the best route to take to minimize her time. She can kayak at 4mph and jog at 7mph. The river is 4 miles wide and the finish line is 6 miles down the beach across from the starting point. What is her best course of action?

Step 1: Sometimes these types of questions have a diagram already provided (lower left). The diagram on the right shows the path the athlete must take. We assume that she will kayak towards some point on the beach where she can then jog towards the finish line.

![Diagram showing the path of the athlete](image)

To minimize time, we must remember that distance = rate • time. So we solve for time as a function of x as:

\[ t(x) = \frac{\sqrt{16 + x^2}}{4} + \frac{6-x}{7} \]

Step 2: Our only constraints are time cannot be negative and \( x \) must be less than 6.

Step 4: Not needed in this example

Step 5: We take the derivative of \( t(x) \) and find the critical numbers.

\[ t'(x) = \frac{2x}{8\sqrt{16 + x^2}} - \frac{1}{7} \]

After setting this equal to zero and solving we get:

\[ x = \pm 2.785 \]

Since \( x \) cannot be negative our only critical number is 2.785 which gives us our minimum time. After plugging this value in to our function, we see that our best time is 1.6778 hours. No other value of \( x \) gives a lower time than this.
EXAMPLE 3:  

Minimize Volume Cost

A box company wants to make a box whose base length is 2 times the base width. The material used to build the top and bottom cost $10 per ft² and the material used to build the sides cost $5 per ft². If the box must have a volume of 50ft³, determine the dimensions that will minimize the cost to build the box.

Step 1: When we say volume we must think 3D. A box has six sides: a top and bottom, and four sides all around.

Step 2: Since we want to minimize the cost which is given as an area that depends on the side, we work our cost function to be:

\[ C = \text{top & bottom} + \text{two sides} + \text{other two sides} \]

\[ C = (10)(2)w^2 + (5)(2)(2w)(h) + (5)(2)wh \]

\[ C = 40w^2 + 30wh \]

Step 3: Our constraint is that the volume is held to be a constant 50 cubic feet so:

\[ 50 = 2w^3h \]

Step 4: We can solve for either h or w but to keep it simple we'll solve for h and plug it back into the cost function:

\[ h = \frac{25}{w^2} \Rightarrow C = 40w^2 + 30w\left(\frac{25}{w^2}\right) \Rightarrow \]

\[ C = 40w^2 + \frac{750}{w} \]

Step 5: Taking the derivative to find the critical numbers we get:

\[ C' = 80w - \frac{750}{w^2} \]

\[ \Rightarrow \]

\[ C' = \frac{80w^3 - 750}{w^2} \]  

Finding our critical numbers from the numerator and denominator we have 0 (which cannot be the answer), and 2.1085. The value of 2.1085 ft is correct and gives us the minimum cost. Using the constraint function we find h to be 5.62 ft. The minimum cost is found to be $533 per box. (Note: A graph can be used as well)
EXAMPLE 4:  \textit{Maximize Volume}

A sheet of cardboard is 3 feet by 4 feet. We want to make a box by cutting equal-sized squares from each corner and folding up the four edges. What should be the dimensions of the box that gives the largest volume?

\textit{Step 1}: Sketch: picture actually cutting the box and then folding up the sides. What would the dimensions be in terms of $X$?

\begin{center}
\begin{tikzpicture}
\draw[->] (0,0) -- (3,0) node[right] {4 feet};
\draw[->] (0,0) -- (0,3) node[above] {3 feet};
\draw (0,0) -- (0,2) -- (3,2) -- (3,0) -- cycle;
\draw (0,0) -- (0,2) -- (1,2);\draw (3,0) -- (3,2) -- (2,2);
\draw (0,0) -- (2,0);\draw (0,2) -- (2,2);
\draw (0,0) -- (0,1);\draw (3,0) -- (3,1);
\node at (1,2) {X};\node at (2,2) {X};\node at (0,1) {X};\node at (3,1) {X};
\end{tikzpicture}
\end{center}

\begin{center}
\begin{tikzpicture}
\draw[->] (0,0) -- (4,0);\draw[->] (0,0) -- (0,3);
\draw (0,0) -- (4,0) -- (4,3) -- (0,3) -- cycle;
\draw (0,0) -- (4,0) -- (4,2);\draw (0,0) -- (0,2);
\node at (2,3) {X};\node at (2,0) {(3-2X)};\node at (4,2) {(4-2X)};
\end{tikzpicture}
\end{center}

\textit{Step 2}: Since we want to maximize volume and we know that $V = l \cdot w \cdot h$, our function is $V = (4 - 2x)(3 - 2x)(x)$ which simplifies to $V = 4x^3 - 14x^2 + 12x$.

\textit{Step 3}: Our only constraint is that $X$ must be less than $\frac{3}{2}$ or 1.5.

\textit{Step 4}: Not needed

\textit{Step 5}: Our derivative is: $V' = 12x^2 - 28x + 12$. Setting equal to zero and then using the quadratic formula to solve we find our critical numbers to be: .566 and 1.77.

Since $X$ must be less than 1.5, we know .566 is our only critical number and check to verify this to be the answer. Our dimensions are $1.87 \times 2.87 \times .566$.

\textit{Note}: To show once again this answer can be found by graphing, the volume function is graphed and the relative max is shown to be at .566.
PRACTICE PROBLEMS

OPTIMIZATION PROBLEM 1:
An open rectangular box with square base is to be made from 48 ft.\(^2\) of cardboard. What dimensions will result in a box with the largest possible volume? What is this volume?

Your work solution:

\[
\text{Volume: } V = \pi r^2 h
\]

OPTIMIZATION PROBLEM 2:
Consider a rectangle of that is constrained to have a perimeter of 20 inches. You must form a cylinder by revolving this rectangle about one of its edges. What dimensions of the rectangle will result in a cylinder of maximum volume? What is this volume?

Sketch Given: \( V = \pi r^2 h \)

Your work solution:

\[
\begin{align*}
\text{Answer: } & \quad \text{Width: } 4 \text{ in.} \\
& \quad \text{Height: } 2 \text{ in.} \\
& \quad \text{Length: } 4 \text{ in.}
\end{align*}
\]